

Hidden Ghost in Massive gravity

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Abstract

The Hessian's determinant for a version of massive gravity given by an infinite expansion of a square root function of the induced metric, vanishes. We show that it allows us to eliminate one of four scalar fields used to generate the graviton mass. This, however, gives rise to the appearance of extra terms in the action with the squared time derivative of the metric, thus signaling that a nonlinear ghost survives. We demonstrate this phenomenon considering a simple system with constraint, which is supposed to reduce the number of physical degrees of freedom, however, we explicitly show how the constraint forces the metric to propagate an extra tachyonic state.

1 Introduction

There is now considerable amount of work on a consistent formulation of massive gravity [1]. The breakdown of diffeomorphism invariance implies that the two helicities ± 2 of the massless graviton would be joined by four degrees to give six degrees of freedom [2]. This would then correspond to five degrees of freedom for the massive graviton in addition to a ghost degree of freedom for the time-like component of broken diffeomorphism. For the Fierz-Pauli choice of the mass-term the ghost degree of freedom decouples in the lowest order [3]. This model for massive gravity can be made consistent breaking diffeomorphism invariance spontaneously by the use of four scalar fields ϕ^A which will be absorbed by the massless graviton [4], [5]. The nonlinear Boulware-Deser ghost decouples up to the third order but reappears at higher orders [6], [7]. It is claimed that for one family of theories with the action given in terms of an infinite expansion of a square root function depending on the induced metric [8], the BD ghost decouples to all orders [9]. The Hessian of such action was shown to vanish, indicating that one degree of freedom might disappear, thus reducing the six naively expected physical degrees of freedom for the gravity and scalar fields to five and hence removing the nonlinear ghost to all orders of perturbation theory [10]. In this note we will investigate whether this really happens or not? The family of actions with vanishing determinant of the Hessian can be simplified by using auxiliary fields [11]. This allows us to explicitly compute the Hessian, establish that it has a zero mode, and then eliminate the corresponding degree of freedom expressing the action in terms of the physical variables. We will show that, however, the process of elimination of the non-physical scalar field induces new terms in the action which depend on the squared time derivatives of the metric, and as a result ghost degree of freedom re-emerges in a nontrivial way. To illustrate this phenomenon we consider a simple theory, where the constraint is introduced with the purpose to reduce the number of degrees of freedom for massive gravity. We show how in this theory the constraint causes the metric itself to propagate an additional tachyonic state.

2 Massive gravity in the simplified formulation

We consider an equivalent formulation of massive gravity [11] to that in reference [8], which avoids using an infinite expansion in terms of the induced metric. The equivalence, however, is only valid on shell. We first define

$$S'_{AB} = e_A^\mu \partial_\mu \phi_B, \quad (1)$$

$$S_{AB} = S'_{AB} - \eta_{AB}, \quad (2)$$

where the sixteen vierbein fields e_A^μ are subject to the ten conditions

$$g^{\mu\nu} = e_A^\mu e^{\nu A} \equiv e_A^\mu e_A^\nu, \quad (3)$$

where $g^{\mu\nu}$ is the inverse metric, which will be imposed through Lagrange multipliers.

The action is given by [11]

$$I = -\frac{1}{2} \int d^4x \sqrt{g} R + \frac{m^2}{2} \int d^4x \sqrt{g} (S^2 - S_{AB} S_{AB}) + \frac{1}{2} \int d^4x \sqrt{g} \tau_{\mu\nu} (g^{\mu\nu} - e_A^\mu e^{\nu A}), \quad (4)$$

where we use the units in which $8\pi G = 1$. One can also add to this action the terms [12]

$$\int d^4x \sqrt{g} \left(c_1 \delta_{A'B'C'}^{ABC} S_A^{A'} S_B^{B'} S_C^{C'} + c_2 \delta_{A'B'C'D'}^{ABCD} S_A^{A'} S_B^{B'} S_C^{C'} S_D^{D'} \right), \quad (5)$$

that do not influence the nature of the ghost decoupling. However, to simplify the calculations we will not consider them in our analysis. Variation of the action with respect to the metric $g^{\mu\nu}$ gives

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \tau'_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tau', \quad (6)$$

where

$$\tau'_{\mu\nu} = \tau_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Lambda, \quad \Lambda = \frac{m^2}{2} (S^2 - S_{AB} S_{AB}), \quad (7)$$

or equivalently,

$$\tau'_{\mu\nu} = R_{\mu\nu}. \quad (8)$$

The variation with respect to e_A^μ leads to

$$\tau_{\mu\nu} e_A^\nu = -m^2 (S_{AC} \partial_\mu \phi_C - S \partial_\mu \phi_A). \quad (9)$$

Multiplying this equation with e_B^μ we have

$$\begin{aligned} e_B^\mu \tau_{\mu\nu} e_A^\nu &= -m^2 (S_{AC} S'_{BC} - S S'_{BA}) \\ &= -m^2 (S_{AC} S_{BC} - S S_{BA} + S_{AB} - \eta_{AB} S). \end{aligned} \quad (10)$$

The symmetry of the tensor $\tau_{\mu\nu}$ implies that the left hand side of the above equation is symmetric with respect to the indices A and B

$$e_B^\mu \tau_{\mu\nu} e_A^\nu = e_B^\nu \tau_{\mu\nu} e_A^\mu, \quad (11)$$

and thus

$$S_{AC} S_{BC} - S S_{BA} + S_{AB} = S_{BC} S_{AC} - S S_{AB} + S_{BA}, \quad (12)$$

or, equivalently

$$(S + 1) (S_{AB} - S_{BA}) = 0. \quad (13)$$

The possible solution

$$S = -1, \quad (14)$$

when combined with equation (9) gives

$$\tau_{\mu\nu} = -m^2 (\partial_\mu \phi_C \partial_\nu \phi_C). \quad (15)$$

This is inconsistent with the vacuum solution [4]:

$$e_A^\mu = \delta_A^\mu, \quad \phi^A = x^A, \quad (16)$$

for which $S = 4$. We thus assume that $S \neq -1$, and hence

$$S_{AB} = S_{BA}. \quad (17)$$

which implies that S'_{AB} is also symmetric. Using the identity

$$S'_{AB} S'_{AC} = e_A^\mu \partial_\mu \phi_B e_A^\nu \partial_\nu \phi_C = g^{\mu\nu} \partial_\mu \phi_B \partial_\nu \phi_C = H_{BC}, \quad (18)$$

$$H_{AB} \equiv \eta_{AB} + \bar{h}_{AB}, \quad (19)$$

we conclude that, since S'_{AB} is symmetric,

$$\begin{aligned} S_{AB} &= S'_{AB} - \eta_{AB} = \sqrt{\eta_{AB} + \bar{h}_{AB}} - \eta_{AB} \\ &= \frac{1}{2} \bar{h}_{AB} - \frac{1}{8} \bar{h}_{AC} \bar{h}_{CB} + \dots \end{aligned} \quad (20)$$

Substitution of this expression in the action (4) leads on shell (i.e. after solving the constraint equations (3)) and (17)) to the infinite expansion given in [8].

3 Vanishing of the Hessian's determinant on shell

We now compute the Hessian of the terms depending on the scalar fields ϕ^A in the action (4), defined as

$$\mathcal{H}_{CD} = \frac{\delta^2 I}{\delta (\partial_0 \phi^C) \delta (\partial_0 \phi^D)}, \quad (21)$$

taking into account that 16 constraint equations

$$g^{\mu\nu} = e_A^\mu e^{\nu A}, \quad (22)$$

$$e_A^\mu \partial_\mu \phi_B = e_B^\mu \partial_\mu \phi_A. \quad (23)$$

which allow us to express e_A^μ in terms of the fields $g^{\mu\nu}$ and $\partial_\mu \phi_A$. First we evaluate the variations

$$\frac{\delta S}{\delta (\partial_0 \phi^C)} = e_C^0 + Y_{AC}^\mu \partial_\mu \phi_A \quad (24)$$

$$\frac{\delta S_{AB}}{\delta (\partial_0 \phi^C)} = e_A^0 \eta_{BC} + Y_{AC}^\mu \partial_\mu \phi_B \quad (25)$$

where

$$Y_{AC}^\mu = \frac{\delta e_A^\mu}{\delta (\partial_0 \phi^C)}. \quad (26)$$

and setting

$$\left[\frac{\delta I}{\delta e_A^\mu} \right]_{\text{onshell}} = 0 \quad (27)$$

which gives the equation

$$S_{AB} \partial_\mu \phi_B = S \partial_\mu \phi_A \quad (28)$$

After some algebra we obtain for the second variation of the action

$$\begin{aligned} \mathcal{H}_{CD} = m^2 [& (Y_{CD}^0 + Y_{DC}^0) S - (Y_{AD}^0 S_{AC} + Y_{AC}^0 S_{AD}) + e_C^0 e_D^0 - g^{00} \eta_{CD} \\ & + (e_C^0 Y_{AD}^\mu + e_D^0 Y_{AC}^\mu) \partial_\mu \phi^A - e_A^0 (Y_{AD}^\mu \partial_\mu \phi_C + Y_{AC}^\mu \partial_\mu \phi_D) \\ & - Y_{AD}^\mu Y_{AC}^\nu \partial_\mu \phi_B \partial_\nu \phi_B + Y_{BD}^\mu Y_{AC}^\nu \partial_\mu \phi_B \partial_\nu \phi_A] \end{aligned} \quad (29)$$

where we have used $g^{00} = e_C^0 e_C^0$. One can only determine the functions Y_{AB}^μ perturbatively. To avoid this inconvenience, we first establish some properties of these functions, and then use them to simplify the expression of \mathcal{H}_{CD} . First, we differentiate $g^{\mu\nu}$ to obtain

$$\begin{aligned} 0 &= \frac{\delta g^{\mu\nu}}{\delta (\partial_0 \phi^C)} = \frac{\delta}{\delta (\partial_0 \phi^C)} (e_A^\mu e_A^\nu) \\ &= Y_{AC}^\mu e_A^\nu + e_A^\mu Y_{AC}^\nu, \end{aligned} \quad (30)$$

and, in particular,

$$e_A^0 Y_{AC}^0 = 0. \quad (31)$$

From (23) we get the relations

$$\begin{aligned} 0 &= \frac{\delta}{\delta (\partial_0 \phi^C)} (e_A^\mu \partial_\mu \phi_B - e_B^\mu \partial_\mu \phi_A) \\ &= Y_{AC}^\mu \partial_\mu \phi_B - Y_{BC}^\mu \partial_\mu \phi_A + e_A^0 \eta_{BC} - e_B^0 \eta_{AC}. \end{aligned} \quad (32)$$

Finally differentiating the identity $S'_{AB} S'_{AB} = g^{\mu\nu} \partial_\mu \phi_A \partial_\nu \phi_A$, we obtain

$$\begin{aligned} 0 &= \frac{\delta}{\delta (\partial_0 \phi^C)} (S'_{AB} S'_{AB} - g^{\mu\nu} \partial_\mu \phi_A \partial_\nu \phi_A) \\ &= 2 (S'_{AB} (Y_{AC}^\mu \partial_\mu \phi_B + e_A^0 \eta_{BC}) - g^{\mu 0} \partial_\mu \phi_C). \end{aligned} \quad (33)$$

Because

$$S'_{AB} e_A^0 \eta_{BC} = g^{0\mu} \partial_\mu \phi_C, \quad (34)$$

this implies that

$$S'_{AB} Y_{AC}^\mu \partial_\mu \phi_B = 0, \quad (35)$$

and the e_A^μ equations of motion (9) then give

$$\begin{aligned} 0 &= S'_{AB} Y_{AC}^\mu \partial_\mu \phi_B = -\frac{1}{m^2} \tau_{\mu\nu} e_A^\nu Y_{AC}^\mu + (S+1) \partial_\mu \phi_A Y_{AC}^\mu \\ &= (S+1) \partial_\mu \phi_A Y_{AC}^\mu, \end{aligned} \quad (36)$$

since

$$\tau_{\mu\nu} e_A^\nu Y_{AC}^\mu = -\tau_{\mu\nu} e_A^\mu Y_{AC}^\nu = 0. \quad (37)$$

Taking into account that $S \neq -1$ it follows from here that

$$\partial_\mu \phi_A Y_{AC}^\mu = 0. \quad (38)$$

This implies that the last term in (29) vanishes. Going back to the expression for the Hessian, we note that

$$-e_A^0 Y_{AD}^\mu \partial_\mu \phi_C = e_A^\mu Y_{AD}^0 \partial_\mu \phi_C = S'_{AC} Y_{AD}^0, \quad (39)$$

The term before the last in (29) is tricky. To simplify it we first vary equation (38) with respect to $\delta(\partial_0 \phi^C)$ to get

$$Y_{DC}^0 + \partial_\mu \phi_A X_{ADC}^\mu = 0 \quad (40)$$

where we have defined

$$X_{ADC}^\mu = \frac{\delta^2 e_A^\mu}{\delta(\partial_0 \phi^C) \delta(\partial_0 \phi^D)} \quad (41)$$

It is clear from the definition of X_{ADC}^μ in equation (41) and the property of Y_{CD}^0 in equation (40) that

$$Y_{CD}^0 = Y_{DC}^0. \quad (42)$$

Next we vary equation (30) to get

$$Y_{AD}^\mu Y_{AC}^\nu + Y_{AC}^\mu Y_{AD}^\nu + e_A^\mu X_{ACD}^\nu + e_A^\nu X_{ACD}^\mu = 0 \quad (43)$$

which implies the identity

$$Y_{AD}^\mu Y_{AC}^\nu \partial_\mu \phi_B \partial_\nu \phi_B = -e_A^\mu X_{ACD}^\nu \partial_\mu \phi_B \partial_\nu \phi_B \quad (44)$$

$$= -S'_{AB} X_{ACD}^\nu \partial_\nu \phi_B \quad (45)$$

$$= (S+1) Y_{CD}^0 \quad (46)$$

Thus finally the Hessian simplifies to

$$\mathcal{H}_{CD} = m^2 (Y_{CD}^0 (S+1) + e_C^0 e_D^0 - g^{00} \eta_{CD}). \quad (47)$$

One can immediately see that \mathcal{H}_{CD} has the null eigenvector e_D^0 ,

$$\mathcal{H}_{CD} e_D^0 = 0. \quad (48)$$

In addition \mathcal{H}_{CD} satisfy the property

$$e_C^i \mathcal{H}_{CDE} e_D^j = m^2 (Y^{ij} (S+1) + g^{i0} g^{j0} - g^{00} g^{ij}), \quad (49)$$

where

$$Y^{ij} = e_C^i Y_{CD}^0 e_D^j = Y^{ji}. \quad (50)$$

To find the other eigenvectors we write

$$e_C^i Y_{CD}^0 = \alpha^i e_D^0 - \beta g^{00} e_D^i. \quad (51)$$

From $e_C^i Y_{CD}^0 e_D^0 = 0$ it follows that

$$\alpha^i = \beta g^{0i}, \quad (52)$$

and hence

$$Y^{ij} = \beta (g^{i0} g^{j0} - g^{00} g^{ij}). \quad (53)$$

Therefore we have

$$(g^{00} e_C^i - g^{0i} e_C^0) Y_{CD}^0 = -\beta g^{00} (g^{00} e_D^i - g^{0i} e_D^0), \quad (54)$$

and the three eigenvectors $(g^{00} e_C^i - g^{0i} e_C^0)$ has the same eigenvalue $-\beta g^{00}$. To find β we write

$$e_C^\mu Y_{CD}^0 e_D^\nu = \beta (g^{\mu 0} g^{\nu 0} - g^{00} g^{\mu\nu}), \quad (55)$$

and after contracting with the inverse metric $g_{\mu\nu}$ one obtains

$$\eta^{CD} Y_{CD}^0 = -3\beta g^{00} \equiv Y = \frac{\delta e_C^0}{\delta (\partial_0 \phi_C)}. \quad (56)$$

Thus we have

$$\mathcal{H}^{\mu\nu} \equiv e_C^\mu \mathcal{H}_{CD} e_D^\nu \quad (57)$$

$$= \mathcal{H} (g^{\mu 0} g^{\nu 0} - g^{00} g^{\mu\nu}) \quad (58)$$

where

$$\mathcal{H} = \beta (S + 1) + 1. \quad (59)$$

The terms with second time derivative of the fields ϕ^A has the form

$$\frac{1}{2} \partial_0 \phi^C \mathcal{H}_{CD} \partial_0 \phi^D. \quad (60)$$

Let us define

$$\phi_C = x_C + \psi_\mu e_C^\mu, \quad (61)$$

which can be inverted to determine ψ_μ in terms of the metric $g^{\mu\nu}$ and the scalars ϕ^A :

$$\psi_\mu = e_\mu^C (\phi_C - x_C), \quad (62)$$

where e_μ^C is the inverse of e_C^μ . Then the term (60) becomes

$$\begin{aligned} & \frac{1}{2} (\mathcal{H}_{00} + 2\mathcal{H}_{0D} e_D^i \partial_0 \psi_i + \partial_0 \psi_\mu e_C^\mu \mathcal{H}_{CD} e_D^\nu \partial_0 \psi_\nu \\ & + 2\psi_\mu \partial_0 e_C^\mu \mathcal{H}_{CD} e_D^\nu \partial_0 \psi_\nu + \psi_\mu \partial_0 e_C^\mu \mathcal{H}_{CD} \partial_0 e_D^\nu \psi_\nu) \end{aligned} \quad (63)$$

Since the projection of $e_C^\mu \mathcal{H}_{CD} e_D^\nu$ is zero if either $\mu = 0$ or $\nu = 0$, the above expression reduces to

$$\frac{1}{2} m^2 \partial_0 \psi_i \mathcal{H}^{ij} \partial_0 \psi_j + \dots \quad (64)$$

There are, however, couplings of ψ_0 to the derivatives of $\partial_0 e_C^0$, e.g.

$$\frac{1}{2} (m^2 \psi_0 \partial_0 e_C^0 \partial_0 e_D^0 (Y_{CD}^0 (S+1) + e_C^0 e_D^0 - g^{00} \eta_{CD})) \quad (65)$$

Note that

$$\partial_0 e_C^0 e_C^0 = \frac{1}{2} \partial_0 g^{00} \quad (66)$$

and, hence, the squared time derivatives of the metric appear signalling that the metric itself can propagate extra degree of freedom. On the other hand the term $\partial_0 e_C^0 \partial_0 e_C^0$ cannot be expressed only in terms of the metric, but will also involve the antisymmetric part of e_C^0 .

Moreover the action contains also the terms linear in $(\partial_0 \phi_C)$ which do not contribute to the Hessian. For example, after integrating by parts we obtain the following linear term

$$3m^2 \partial_0 (\sqrt{g} e_C^0) e_C^0 \psi_0, \quad (67)$$

which after eliminating the ψ_0 as auxiliary field gives complicated contributions which depend on the time derivatives of the metric. We conclude that although the determinant of the Hessian vanishes, eliminating the zero mode state ψ_0 introduces squared time derivatives of the metric signalling the propagation of the ghost states.

4 Hamiltonian analysis off shell

Having proved the equivalence of the on shell formulation of massive gravity to the square root action, one should be able to see that the metric acquires second time derivatives without having to go on shell. We again start with the action (4) but now treat e_A^μ as independent variable. To do the Hamiltonian analysis, we have to solve for the equations for conjugated momenta in terms of the fields ϕ_A . From the previous analysis, it is clear that it is more convenient to use the variables ψ_μ , defined via,

$$\phi_A = x_A + e_A^\mu \psi_\mu, \quad (68)$$

instead of ϕ_A . The part of the action containing scalars is then

$$\begin{aligned} \frac{m^2}{2} \sqrt{g} [& - (g^{\mu\nu} g^{\kappa\lambda} \partial_\mu \psi_\kappa \partial_\nu \psi_\lambda + 2g^{\mu\nu} e_A^\kappa \partial_\nu e_A^\lambda \partial_\mu \psi_\kappa \psi_\lambda + g^{\mu\nu} \partial_\mu e_A^\kappa \partial_\nu e_A^\lambda \psi_\kappa \psi_\lambda) \\ & + (g^{\mu\nu} \partial_\mu \psi_\nu + e_A^\mu \partial_\mu e_A^\nu \psi_\nu)^2 - 6 (g^{\mu\nu} \partial_\mu \psi_\nu + e_A^\mu \partial_\mu e_A^\nu \psi_\nu) \\ & - 2g^{\mu\nu} \delta_\mu^A (\partial_\nu e_A^\kappa \psi_\kappa + e_A^\kappa \partial_\nu \psi_\kappa) + 2e_A^\kappa \delta_\kappa^A (g^{\mu\nu} \partial_\mu \psi_\nu + e_A^\mu \partial_\mu e_A^\nu \psi_\nu) \\ & + e_A^\mu e_B^\nu (\delta_\mu^A \delta_\nu^B - \eta^{AB} \eta_{\mu\nu}) - 6e_A^\mu \delta_\mu^A + 12]. \end{aligned} \quad (69)$$

The term dependent on the second derivative of ψ_μ is

$$(g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\nu}g^{\kappa\lambda}) \partial_\mu \psi_\kappa \partial_\nu \psi_\lambda, \quad (70)$$

where $(\partial_0 \psi_0)^2$ drops out. The problem now is that there remains a term linear in $\partial_0 \psi_0$ which is

$$m^2 \sqrt{g} \partial_0 \psi_0 \left[-3g^{00} + (g^{00}e_A^i - g^{0i}e_A^0) \left(\partial_i e_A^0 \psi_0 + \partial_i e_A^j \psi_j \right) + (e_A^i g^{00} - g^{i0} e_A^0) \delta_i^A \right]. \quad (71)$$

It is clear that the conjugated momentum for the field ψ_0 does not allow to solve for $\partial_0 \psi_0$, and therefore we first integrate the time derivative of ψ_0 by parts

$$\begin{aligned} & -m^2 \psi_0 \left[\psi_j \partial_0 \left(\sqrt{g} (g^{00}e_A^i - g^{0i}e_A^0) \partial_i e_A^j \right) + \sqrt{g} \partial_0 \psi_j (g^{00}e_A^i - g^{0i}e_A^0) \partial_i e_A^j \right. \\ & \quad \left. + \frac{1}{2} \psi_0 \partial_0 (\sqrt{g} (g^{00}e_A^i - g^{0i}e_A^0) \partial_i e_A^0) \right. \\ & \quad \left. - 3\partial_0 (\sqrt{g} g^{00}) + \partial_0 (\sqrt{g} (e_A^i g^{00} - g^{i0} e_A^0) \delta_i^A) \right] \end{aligned} \quad (72)$$

One can now solve the equations

$$\begin{aligned} \frac{1}{m^2 \sqrt{g}} \pi^i &= ((g^{0i}g^{0j} - g^{00}g^{ij}) \partial_0 \psi_j + (g^{0i}g^{jk} - g^{0j}g^{ik}) \partial_j \psi_k \\ & \quad + (g^{0j}e_A^0 - g^{00}e_A^j) (\partial_j e_A^i \psi_0 + \delta_j^i \partial_0 e_A^\nu \psi_\nu) \\ & \quad + (g^{0i}e_A^j - g^{0j}e_A^i) \partial_j e_A^\nu \psi_\nu + (e_A^\mu g^{i0} - e_A^i g^{\mu 0}) \delta_\mu^A - 3g^{0i}), \end{aligned} \quad (73)$$

to express $\partial_0 \psi_i$ in terms of the conjugated momentum π_i . The action then becomes

$$\frac{1}{2} m^2 \sqrt{g} \left[\frac{1}{m^4 g} \pi^i M_{ij}^{-1} \pi^j - B^i M_{ij}^{-1} B^j \right], \quad (74)$$

where

$$\begin{aligned} M^{ij} &= (g^{0i}g^{0j} - g^{00}g^{ij}) \\ B^i &= (g^{0i}g^{kl} - g^{0k}g^{il}) \partial_k \psi_l + (g^{0k}e_A^0 - g^{00}e_A^k) (\partial_k e_A^i \psi_0 + \delta_k^i \partial_0 e_A^\nu \psi_\nu) \\ & \quad + (g^{0i}e_A^k - g^{0k}e_A^i) \partial_k e_A^\nu \psi_\nu + (e_A^\mu g^{i0} - e_A^i g^{\mu 0}) \delta_\mu^A - 3g^{0i}. \end{aligned} \quad (75)$$

The ψ_0 dependent terms take the form

$$\frac{1}{2} m^2 \sqrt{g} (\psi_0 \mathcal{A} \psi_0 + \mathcal{B} \psi_0), \quad (77)$$

where

$$\begin{aligned} \mathcal{A} &= -\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} (g^{0i}g^{0j} - g^{00}g^{ij}) \partial_j) - \frac{1}{\sqrt{g}} \partial_0 (\sqrt{g} (g^{00}e_A^i - g^{0i}e_A^0) \partial_i e_A^0) \\ & \quad + (e_A^\mu \partial_\mu e_A^0)^2 + 2 (g^{0i}e_A^j - g^{ji}e_A^0) \partial_j e_A^0 \partial_i - g^{\mu\nu} \partial_\mu e_A^0 \partial_\nu e_A^0 - C^i M_{ij}^{-1} C^j, \end{aligned} \quad (78)$$

is a second order differential operator with respect to the spacial coordinates, and we defined

$$C^i = (g^{0k}e_A^0 - g^{00}e_A^k) (\partial_k e_A^i + \delta_k^i \partial_0 e_A^0) + (g^{0i}e_A^k - g^{0k}e_A^i) \partial_k e_A^0. \quad (79)$$

Similarly

$$\begin{aligned} \mathcal{B} = & 2(e_A^\mu \partial_\mu e_A^0) (e_B^\nu \partial_\nu e_B^i) \psi_i - \frac{2}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{0i} e_A^\mu \partial_\mu e_A^j \psi_j \right) + 2g^{ij} (e_A^\mu \partial_\mu e_A^0) \partial_i \psi_j \\ & + \frac{2}{\sqrt{g}} \partial_i \left(\sqrt{g} (g^{ij} g^{0k} - g^{kj} g^{0i}) \partial_j \psi_k \right) + \frac{2}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{\mu i} e_A^0 \partial_\mu e_A^j \psi_j \right) - 2g^{\mu i} e_A^j \partial_\mu e_A^0 \partial_i \psi_j \\ & - 2g^{\mu\nu} \partial_\mu e_A^0 \partial_\nu e_A^i \psi_i + \frac{6}{\sqrt{g}} \partial_i (\sqrt{g} g^{0i}) - 6e_A^\mu \partial_\mu e_A^0 + 2\delta_\mu^A (e_A^\mu e_B^\nu - g^{\mu\nu} \eta_{AB}) \partial_\nu e_B^0 \\ & - 2(g^{0k}e_A^0 - g^{00}e_A^k) (\partial_k e_A^i + \delta_k^i \partial_0 e_A^0) M_{ij}^{-1} \bar{B}^j - \frac{2}{\sqrt{g}} \delta_j^A \partial_i \left(\sqrt{g} (e_A^j g^{0i} - e_A^0 g^{ji}) \right) \\ & + \frac{6}{\sqrt{g}} \partial_0 (\sqrt{g} g^{00}) - \frac{2}{\sqrt{g}} \partial_0 (\sqrt{g} (e_A^i g^{00} - e_A^0 g^{i0}) \delta_i^A) \\ & - \frac{2}{\sqrt{g}} \psi_j \partial_0 \left(\sqrt{g} (g^{00} e_A^i - g^{0i} e_A^0) \partial_i e_A^j \right), \end{aligned} \quad (80)$$

where

$$\begin{aligned} \bar{B}^i = & (g^{0i} g^{kl} - g^{0k} g^{il}) \partial_k \psi_l + (g^{0i} e_A^0 - g^{00} e_A^i) \partial_0 e_A^j \psi_j \\ & + (g^{0i} e_A^k - g^{0k} e_A^i) \partial_k e_A^j \psi_j + (e_A^\mu g^{0i} - e_A^i g^{0\mu}) \delta_\mu^A - 3g^{0i}. \end{aligned} \quad (81)$$

All remaining terms would depend only on ψ_i and their spacial derivatives and the momenta π^i . Eliminating the ψ_0 results to

$$- \frac{1}{8} m^2 \sqrt{g} \mathcal{B} \mathcal{A}^{-1} \mathcal{B} \quad (82)$$

and the full action becomes

$$\begin{aligned} \frac{m^2}{2} \sqrt{g} \left[& (g^{ij} g^{kl} - g^{ik} g^{jl}) \partial_i \psi_j \partial_k \psi_l + (e_A^\mu \partial_\mu e_A^i) (e_B^\nu \partial_\nu e_B^j) \psi_i \psi_j \right. \\ & + 2(g^{ij} e_A^\mu - g^{i\mu} e_A^j) \partial_\mu e_A^k \partial_i \psi_j \psi_k - g^{\mu\nu} \partial_\mu e_A^i \partial_\nu e_A^j \psi_i \psi_j \\ & + e_A^\mu e_B^\nu (\delta_\mu^A \delta_\nu^B - \eta^{AB} \eta_{\mu\nu}) - 6e_A^\mu \delta_\mu^A + 12 \\ & + 2\delta_\mu^A (e_A^\mu e_B^\nu - g^{\mu\nu} \eta_{AB}) \partial_\nu e_B^i \psi_i + 2\delta_\mu^A (e_A^\mu g^{ji} - e_A^j g^{\mu i}) \partial_i \psi_j \\ & \left. - 6g^{ij} \partial_i \psi_j - 6e_A^\mu \partial_\mu e_A^i \psi_i + \frac{1}{m^4 g} \pi^i M_{ij}^{-1} \pi^j - \bar{B}^i M_{ij}^{-1} \bar{B}^j - \frac{1}{4} \mathcal{B} \mathcal{A}^{-1} \mathcal{B} \right], \end{aligned} \quad (83)$$

where

$$(M^{-1})_{ij} = \frac{1}{g^{00}} \left(g_{ij}^{-1} + \frac{1}{g^{00} - g^{0k} g^{0l} g_{kl}^{-1}} g^{0k} g^{0l} g_{ik}^{-1} g_{jl}^{-1} \right), \quad (84)$$

and g_{ij}^{-1} is the inverse of the three dimensional metric g^{ij} , $g^{ik}g_{kj}^{-1} = \delta_j^i$.

We can easily see the metric dependent part of this action by setting $\psi_i = 0$:

$$\frac{m^2}{2}\sqrt{g}\left(e_A^\mu e_B^\nu(\delta_\mu^A \delta_\nu^B - \eta^{AB}\eta_{\mu\nu}) - 6e_A^\mu \delta_\mu^A + 12 - \overline{B}_0^i M_{ij}^{-1} \overline{B}_0^j - \frac{1}{4}\mathcal{B}_0 \mathcal{A}^{-1} \mathcal{B}_0\right), \quad (85)$$

where

$$\begin{aligned} \mathcal{B}_0 = & \frac{6}{\sqrt{g}}\partial_i(\sqrt{g}g^{0i}) - 6e_A^\mu \partial_\mu e_A^0 + 2\delta_\mu^A(e_A^\mu e_B^\nu - g^{\mu\nu}\eta_{AB})\partial_\nu e_B^0 \\ & - 2(g^{0k}e_A^0 - g^{00}e_A^k)(\partial_k e_A^i + \delta_k^i \partial_0 e_A^0)M_{ij}^{-1}\overline{B}_0^j - \frac{2}{\sqrt{g}}\delta_j^A \partial_i(\sqrt{g}(e_A^j g^{0i} - e_A^0 g^{ji})) \\ & - \frac{2}{\sqrt{g}}\delta_j^A \partial_i(\sqrt{g}(e_A^j g^{0i} - e_A^0 g^{ji})) + \frac{6}{\sqrt{g}}\partial_0(\sqrt{g}g^{00}) - \frac{2}{\sqrt{g}}\partial_0(\sqrt{g}(e_A^i g^{00} - e_A^0 g^{i0})\delta_i^A), \end{aligned} \quad (86)$$

and

$$\overline{B}_0^i = (e_A^\mu g^{0i} - e_A^i g^{0\mu})\delta_\mu^A - 3g^{0i}. \quad (87)$$

The action is non-local because of the operator \mathcal{A}^{-1} . We can simplify it further by assuming that e_A^μ depend only on time. Then we have

$$\begin{aligned} \mathcal{A} = & (e_A^0 \partial_0 e_A^0)^2 - g^{00}\partial_0 e_A^0 \partial_0 e_A^0 - (g^{0i}e_A^0 - g^{00}e_A^i)M_{ij}^{-1}(g^{0j}e_A^0 - g^{00}e_A^j) \\ = & \frac{1}{4}(\partial_0 g^{00})^2 - g^{00}\partial_0 e_A^0 \partial_0 e_A^0 + 3g^{00}, \end{aligned} \quad (88)$$

and

$$\begin{aligned} \mathcal{B}_0 = & -3\partial_0 g^{00} + \delta_\mu^A(e_A^\mu \partial_0 g^{00} - 2g^{\mu 0}\partial_0 e_A^0) + \frac{6}{\sqrt{g}}\partial_0(\sqrt{g}g^{00}) \\ & - (g^{0i}\partial_0 g^{00} - 2g^{00}e_A^i \partial_0 e_A^0)M_{ij}^{-1}\left((e_B^\mu g^{0j} - e_B^j g^{0\mu})\delta_\mu^B - 3g^{0j}\right) \\ & - \frac{2}{\sqrt{g}}\partial_0(\sqrt{g}(e_A^i g^{00} - e_A^0 g^{i0})\delta_i^A). \end{aligned} \quad (89)$$

It is clear that in the linearized approximation, where $g^{00} = 1 + h^{00}$, the field h^{00} acquires terms with second time derivatives, and thus become ghost like.

The conclusion is that, the determinant of the Hessian indeed vanishes, indicating that one of the scalar fields, in this case ψ_0 , could be eliminated completely, but this turns out to be a curse rather than a blessing: new contributions to the metric appear which force the ghost modes in gravity, previously protected by diffeomorphism invariance, to propagate.

5 Constraints and induced degrees of freedom

We now illustrate the conclusions reached in the last two sections by considering the simple example of massive gravity where we explicitly impose a constraint

on the system with the purpose to decrease the number degrees of freedom in the system. However, contrary to our expectations we discover that the previously “silent” gravitational degrees of freedom are forced by constraint to become propagating. This mimics the situation with vanishing Hessian’s determinant.

Let us consider a massive gravity with four scalars ϕ^A [5] subject to the constraint

$$H = 4, \quad \Rightarrow \bar{h} = 0, \quad (90)$$

where $H^{AB} = g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B$, and $H = H^{AB} \eta_{AB} = 4 + \bar{h}$.

We will now examine whether the above constraint really reduces the original six degrees of freedom to the five ones, required for the massive graviton.

We will analyze this system in different ways. First, we implement the condition $\bar{h} = 0$ in the action expanded to the second order in perturbations

$$S = \frac{1}{8} \int d^4x \left[\partial^C h_B^A \partial_C h_A^B - 2\partial^C h_C^A \partial_D h_A^D + 2\partial^C h_C^A \partial_A h - \partial_A h \partial^A h - m^2 \bar{h}_B^A \bar{h}_A^B \right]. \quad (91)$$

Taking into account that $\bar{h}_{AB} = h_{AB} + \partial_A \chi_B + \partial_B \chi_A$ the last term here becomes

$$- \frac{m^2}{8} (h^{AB} h_{AB} + 4\partial^A \chi^B h_{AB} + 2\partial^A \chi^B \partial_A \chi_B + 2\partial^A \chi^B \partial_B \chi_A). \quad (92)$$

Using the constraint $\bar{h} = h + 2\partial^A \chi_A = 0$ we can write

$$\chi_A = \hat{\chi}_A - \frac{1}{2\partial^2} \partial_A h, \quad (93)$$

where $\partial^A \hat{\chi}_A = 0$.

To determine the propagators, we fix the gauge by choosing the following gauge conditions,

$$G = H - 4 = 0, \quad (94)$$

$$G^A = \partial^B h_B^A - \frac{1}{2} \partial_A h - m^2 \chi^A = 0, \quad (95)$$

which have to be imposed via corresponding delta functions in the path integral

$$\int [Dh^{\mu\nu}] [D\chi^A] \delta(\bar{h}) \delta\left(\partial^B h_B^A - \frac{1}{2} \partial^A h - m^2 \chi^A\right) e^{-S}. \quad (96)$$

Next we add to the action the gauge fixing term

$$\frac{1}{4\alpha} \left(\partial_A h^{AB} - \frac{1}{2} \partial^B h - m^2 \chi^B \right)^2, \quad (97)$$

and to simplify the expressions we set $\alpha = 1$ to cancel the cross terms. The

action then reduces to:

$$\begin{aligned}
S &= \frac{1}{8} \int d^4x \left[\partial^C h_B^A \partial_C h_A^B - \frac{1}{2} \partial_A h \partial^A h - m^2 h_B^A h_A^B \right. \\
&\quad \left. + 2m^4 \chi^B \chi_B - 2m^2 h \partial^B \chi_B - 2m^2 \partial^A \chi^B \partial_A \chi_B - 2m^2 \partial^A \chi^B \partial_B \chi_A \right] \\
&= \frac{1}{8} \int d^4x \left(-h_B^A (\partial^2 + m^2) h_A^B + \frac{1}{2} h (\partial^2 + m^2) h + 2m^2 \chi^B (\partial^2 + m^2) \chi_B \right),
\end{aligned} \tag{98}$$

where we have taken into account that $\bar{h} = h + 2\partial^A \chi_A = 0$. Substituting $\partial^A \chi_A = -\frac{1}{2}h$ gives

$$2m^2 \chi^B (\partial^2 + m^2) \chi_B = 2m^2 \hat{\chi}^B (\partial^2 + m^2) \hat{\chi}^B - \frac{1}{2} h (\partial^2 + m^2) \frac{m^2}{\partial^2} h, \tag{99}$$

and finally the action reduces to

$$\begin{aligned}
S &= \frac{1}{8} \int d^4x \left(-h_B^A (\partial^2 + m^2) h_A^B + \frac{1}{2} h (\partial^2 + m^2) \left(1 - \frac{m^2}{\partial^2} \right) h \right. \\
&\quad \left. + 2m^2 \hat{\chi}^B (\partial^2 + m^2) \hat{\chi}^B \right).
\end{aligned} \tag{100}$$

Isolating the traceless part of h_{AB} ,

$$h_{AB} = h_{AB}^T + \frac{1}{4} \eta_{AB} h, \tag{101}$$

we obtain for the trace part the following action

$$\begin{aligned}
& -\frac{1}{4} h (\partial^2 + m^2) h + \frac{1}{2} h (\partial^2 + m^2) h - \frac{1}{2} h (\partial^2 + m^2) \frac{m^2}{\partial^2} h \\
&= \frac{1}{4} h (\partial^2 + m^2) \left(1 - \frac{2m^2}{\partial^2} \right) h,
\end{aligned} \tag{102}$$

which clearly contains a tachyonic mode of mass $-2m^2$. In fact, the scalar mode has a propagator

$$\frac{p^2}{(p^2 - m^2)(p^2 + 2m^2)} = \frac{1}{3} \left(\frac{1}{p^2 - m^2} + \frac{2}{p^2 + 2m^2} \right), \tag{103}$$

which represents a combination of a physical spin zero state of mass m and a spin zero *tachyonic state* of mass $m\sqrt{2}$. Together with two degrees of freedom of originally massless graviton and two degrees of freedom induced by a vector part of scalar fields this makes six degrees of freedom in total. Thus the imposed constraint did not reduce the number of the original degrees of freedom but, instead of that, forced the originally “silent” metric components to propagate.

Now we will show how to arrive to the same conclusion by enforcing the constraint $\bar{h} = 0$ through a Lagrange multiplier in the action

$$S = -\frac{1}{2} \int d^4x \sqrt{g} R(g) + \frac{m^2}{8} \int d^4x \sqrt{g} (4 - H_{AB} H^{AB} + \lambda (H - 4)). \tag{104}$$

Varying with respect to $\delta g^{\mu\nu}$ we obtain

$$\begin{aligned} \frac{1}{2} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) &= -\frac{m^2}{16} g_{\mu\nu} (4 - H_{AB} H^{AB} + \lambda(H - 4)) \\ &\quad - \frac{m^2}{4} H^{AB} \partial_\mu \phi_A \partial_\nu \phi_B + \frac{m^2}{8} \partial_\mu \phi^A \partial_\nu \phi^A \lambda, \end{aligned} \quad (105)$$

while the λ equation gives

$$H = 4, \quad (106)$$

and the ϕ^A equations are

$$\frac{2m^2}{\sqrt{g}} \partial_\mu \left[\sqrt{g} (-2g^{\mu\nu} \partial_\nu \phi^B H_{AB} + g^{\mu\nu} \partial_\nu \phi^A \lambda) \right] = 0. \quad (107)$$

In the linearized approximation (with $\lambda = 2 + \bar{\lambda}$ to cancel zero order term) the above two equations become:

$$\begin{aligned} 0 &= (\partial^2 + m^2) h_{AB} - \partial_A \partial^C h_{CB} - \partial_B \partial^C h_{AC} \\ &\quad + \eta_{AB} (\partial^C \partial^D h_{CD} - \partial^2 h) + m^2 (\partial_A \chi_B + \partial_B \chi_A) - \frac{1}{2} m^2 \eta_{AB} \bar{\lambda}, \end{aligned} \quad (108)$$

and

$$\begin{aligned} 0 &= \partial_\mu \left[\left(1 - \frac{h}{2} \right) (-2\eta^{\mu\nu} - 2h^{\mu\nu}) (\delta_\nu^B + \partial_\nu \chi^B) (\eta_{AB} + \bar{h}_{AB}) \right. \\ &\quad \left. + \left(1 - \frac{h}{2} \right) (\eta^{\mu\nu} + h^{\mu\nu}) (\delta_\nu^A + \partial_\nu \chi^A) (2 + \bar{\lambda}) \right]. \end{aligned} \quad (109)$$

Equation (109) simplifies to

$$\partial_A \bar{\lambda} - 2\partial^\mu \bar{h}_{\mu A} = 0. \quad (110)$$

Using the gauge fixing condition

$$\partial^B h_B^A - \frac{1}{2} \partial_A h - m^2 \chi^A = 0, \quad (111)$$

we obtain

$$\begin{aligned} \partial_A \bar{\lambda} &= 2\partial_\mu (h_{\mu A} + \partial_\mu \chi_A + \partial_A \chi_\mu) \\ &= 2(\partial^2 + m^2) \chi_A. \end{aligned} \quad (112)$$

On the other hand, the curvature equation (108) taking into account the gauge condition, gives

$$(\partial^2 + m^2) \left(h_{AB} - \frac{1}{2} \eta_{AB} h \right) = \frac{1}{2} m^2 \eta_{AB} \bar{\lambda}, \quad (113)$$

which is consistent with the divergence of equation (112). Trace of equation (113) leads to

$$-(\partial^2 + m^2)h = 2m^2\bar{\lambda}. \quad (114)$$

The χ_A equation can be written as

$$(\partial^2 + m^2)\left(\hat{\chi}_A - \frac{1}{2}\frac{\partial_A}{\partial^2}h\right) = \frac{1}{2}\partial_A\bar{\lambda}, \quad (115)$$

which implies that

$$\begin{aligned} (\partial^2 + m^2)\hat{\chi}_A &= 0, \\ \bar{\lambda} &= -\frac{\partial^2 + m^2}{\partial^2}h = -\frac{1}{2m^2}(\partial^2 + m^2)h. \end{aligned} \quad (116)$$

Thus

$$(\partial^2 + m^2)\left(1 - \frac{2m^2}{\partial^2}\right)h = 0, \quad (117)$$

which is the same as equation of motion derived from equation (102). Thus we confirm the previous result.

Finally and most directly, the tachyonic mode can be seen if we consider the equations of motion of linearized massive gravity [5], taking into account that

$$\bar{h} = 0, \quad (118)$$

which then take the following form

$$\partial^2\bar{h}_\nu^\mu - \partial_\nu\partial^\rho\bar{h}_\rho^\mu - \partial^\mu\partial_\rho\bar{h}_\nu^\rho + \frac{1}{2}\delta_\nu^\mu\partial^\rho\partial_\sigma\bar{h}_\rho^\sigma + m^2\bar{h}_\nu^\mu = 0 \quad (119)$$

Substituting into these equations the following ansatz

$$\bar{h}_i^0 = \frac{2}{3m^2 + 2\Delta}\partial_i\partial^0\bar{h}_0^0 \quad (120)$$

$$\bar{h}_j^i = -\frac{(\delta_j^i m^2 + 2\partial_j\partial_i)}{3m^2 + 2\Delta}\bar{h}_0^0 \quad (121)$$

one can verify that equations (118) and (119) are both satisfied provided that

$$(\partial_0^2 - \Delta - 2m^2)\bar{h}_0^0 = 0 \quad (122)$$

One can check that all other components of \bar{h}_ν^μ satisfy the same equation, implying that the tachyonic mode with mass $-2m^2$ is propagating.

6 Conclusions

In this paper we have studied the action for massive gravity which is claimed to be ghost free [9] [10]. We have shown that in this theory the Hessian for

the scalar fields, used to generate the graviton mass, has a vanishing determinant, thus implying that at least one of them can be eliminated. However, this does not reduce the number of degrees of freedom of the whole system because the originally silent component of the metric starts to propagate. We have argued then that this seems to be a general behavior of massive gravity, whenever a constraint is imposed to reduce the number of degrees of freedom.

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